

# Entanglement and dephasing of quantum dissipative systems

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The von Neumann entropy of various quantum dissipative models is calculated in order to discuss the entanglement properties of these systems. First, integrable quantum dissipative models are discussed, i.e., the quantum Brownian motion and the quantum harmonic oscillator. In case of the free particle, the related entanglement of formation shows no non-analyticity. In case of the dissipative harmonic oscillator, there is a non-analyticity at the transition of underdamped to overdamped oscillations. We argue that this might be a general property of dissipative systems. We show that similar features arise in the dissipative two level system and study different regimes using sub-Ohmic, Ohmic and super-Ohmic baths, within a scaling approach.

## I. INTRODUCTION

The operation of a quantum computer requires a careful control of the interaction between the system and its macroscopic environment. The resulting entanglement between the system's degrees of freedom and the reservoir has been a recurrent topic since the formulation of quantum mechanics, as it is relevant to the analysis of the measurement process [1, 2]. The loss of coherence due to the interaction of a quantum system and its environment was also extensively studied due to its importance to Macroscopic Quantum Tunneling and related effects [3]. Theoretical research on Macroscopic Quantum Tunneling lead, among other results, to the formulation of a canonical model for the analysis of a quantum system interacting with a macroscopic environment, the so called Caldeira-Leggett model [4]. It can be shown that this harmonic model (see below) describes correctly the low energy features of a system which, in the classical limit, undergoes ohmic dissipation (linear friction). It can be extended to systems with more complicated, non linear, dissipative properties (usually called sub-Ohmic and super-Ohmic, see below) [5, 6].

In relation to the ongoing research on entanglement, a recent interesting development is the analysis of the enhancement of entanglement in a system near a quantum critical point [7, 8]. The original systems under study were the transverse Ising model and the XY model, but also other models which exhibit a quantum phase transition were later investigated in this direction, as e.g. the Lipkin-Meshkov-Glick model [9, 10].

A connection between previous research on Macroscopic Quantum Tunneling and entanglement near quantum critical points is starting to emerge [11]. It is interesting to remark that some of the simplest systems which show a non trivial quantum critical point is the dissipative two level system [5] and related models, like the Kondo model [12]. Similar models describe quantum fluctuations in Josephson junctions [13] or tunneling between Luttinger liquids [14]. It is already known that, even for the ground state of simple models like the dissipative harmonic oscillator non-trivial entanglement properties can be expected as was already commented on in

Ref. [15]. Entanglement energetics at zero temperature was investigated in Ref. [16]. A number of properties of the entanglement in the Caldeira-Leggett model and related models remain, however, unexplored.

The models studied here describe a quantum system characterized by a small number of degrees of freedom coupled to a macroscopic reservoir. These models show a crossover between different regimes, or even exhibit a quantum critical point. As this behavior is induced by the presence of a reservoir with a large number of degrees of freedom, they can also be considered as a model of dephasing and loss of quantum coherence. It is worth noting that there is a close connection between models describing impurities coupled to a reservoir, and strongly correlated systems near a quantum critical point, as evidenced by Dynamical Mean Field Theory [17]. In the limit of large coordination, the properties of an homogeneous system can be reduced to those of an impurity interacting with an appropriately chosen reservoir. Hence, in the limit of large coordination the entanglement between the quantum system and the reservoir near a phase transition can be mapped onto the entanglement which develops in an homogeneous system near a quantum critical point.

The measure of entanglement used in the original papers is the concurrence introduced by Woiters [18]. Alternatively, the von Neuman entropy of macroscopic (contiguous) subsystems can be used [19]. A non-local measure of entanglement was employed in the study of the Affleck-Kennedy-Lieb-Tasaki (AKLT) model [20, 21].

Our previous work [11] showed that the main non-analyticity of the concurrence arises at the transition of coherent to incoherent tunneling. At the actual quantum phase transition, we only found a much weaker non-analyticity associated with the existence of the Kosterlitz-Thouless weakly non analytical features. In the same way, the phase transition of the transverse Ising model discussed in Refs. [7, 8] can be interpreted as a transition where coherence is lost due to the emergence of a localized state at the transition. We thus assume that the loss of coherence might be more important to see non-analyticities in the entanglement of a system than the actual phase transition.

In the first part of this article, we test our assumption using two integrable quantum dissipative models, the dissipative free particle - that is, the Caldeira-Leggett model - and the dissipative quantum harmonic oscillator. These models do not exhibit a quantum phase transition, but, in the latter case there is a transition from underdamped to overdamped oscillations at some critical coupling strength. As measure of entanglement we use the von Neuman entropy of the subsystem, defined using the reduced density matrix,  $\rho_A$ , obtained by tracing out the bath degrees of freedom of the ground state:

$$E(\psi) = -\text{Tr}(\rho_A \ln \rho_A) \quad , \quad \rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|) \quad (1)$$

In the second part of the paper, we make the same analysis for the spin-boson model on the basis of a scaling approach for the free energy. For super-Ohmic baths, the model shows no phase transition whereas for Ohmic and sub-Ohmic baths, there is a transition from localized to non-localized behavior. Again, we focus the discussion on the transition from coherent to incoherent oscillation which exists for Ohmic dissipation, but is also present for certain non-Ohmic environments.

It is finally worth noting that a more mathematical analysis of some problems related to the entanglement in the dissipative harmonic oscillator can be found in Refs. [22, 23]. Sub-Ohmic environments may be relevant to the description of Gaussian effects in qubits coupled to external environments, see [24, 25].

## II. EXACTLY SOLVABLE DISSIPATIVE SYSTEMS

Modeling the environment by a set of harmonic oscillators [4], the general integrable model is described by the following Hamiltonian:

$$H = \frac{p^2}{2} + \frac{\omega_0^2}{2} q^2 + \sum_{\alpha} \left( \frac{p_{\alpha}^2}{2} + \frac{1}{2} \omega_{\alpha}^2 \left( x_{\alpha} - \frac{\lambda_{\alpha}}{\omega_{\alpha}^2} q \right)^2 \right) \quad (2)$$

The operators obey the canonical commutation relations which read ( $\hbar = 1$ )

$$[q, p] = i \quad , \quad [x_{\alpha}, p_{\alpha'}] = i \delta_{\alpha, \alpha'} \quad . \quad (3)$$

The coupling of the system to the bath is completely determined by the spectral function

$$J(\omega) = \frac{\pi}{2} \sum_{\alpha} \frac{\lambda_{\alpha}^2}{\omega_{\alpha}} \quad (4)$$

In the following, we will consider a Ohmic bath with  $J(\omega) = \eta\omega$  for  $\omega \ll \omega_c$  and  $J(\omega) = 0$  for  $\omega \gg \omega_c$ ,  $\omega_c$  being the cutoff frequency.

### A. Caldeira-Leggett model

Let us first consider the free dissipative particle, i.e., we set  $\omega_0 = 0$ . The model was introduced by Caldeira and

Leggett and further investigated by Hakim and Ambegaokar [26, 27]. The latter authors obtained the reduced density matrix via diagonalization of the Hamiltonian:

$$\langle x | \rho_A | x' \rangle = e^{-a(x-x')^2} / L \quad , \quad a = \frac{1}{4\pi} \ln \left( 1 + \frac{\omega_c^2}{\eta^2} \right) \quad (5)$$

where  $\eta$  denotes the friction coefficient and  $\omega_c$  is the cut-off frequency of the bath. Furthermore,  $L \rightarrow \infty$  denotes the system size and in contrary to the use of Eq. 5 in Ref. [27], here the normalization is crucial to assure  $\text{Tr} \rho_A = 1$ .

In order to calculate the entropy of the system, we Taylor expand the logarithm:

$$\ln \rho_A = - \sum_{n=1} \frac{(1 - \rho_A)^n}{n} = - \sum_{n=1} \frac{1}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k \rho_A^k \quad (6)$$

Further we have

$$\langle x | \rho_A^k | x' \rangle = \sqrt{\frac{\pi}{a}}^{k-1} \sqrt{\frac{1}{k}} e^{-\frac{a}{k}(x-x')^2} / L^k \quad (7)$$

proved by induction. With the identity

$$\sqrt{\frac{1}{k}} = \frac{1}{\sqrt{\pi}} \int dx e^{-kx^2} \quad (8)$$

we thus obtain for the specific entropy (for general dimension  $d$ )

$$S = \frac{d}{2} (\ln(aL^2) + 1 - \ln \pi) \quad . \quad (9)$$

Comparing the above result with the entropy of a particle in a canonical ensemble, we identify  $a \sim \lambda^{-2} \propto T$  with  $\lambda$  denoting the thermal de Broglie wavelength and  $T$  the temperature of the canonical ensemble.

Notice that the entropy of a free dissipative particle shows no non-analyticity.

### B. Dissipative harmonic oscillator

We now include the harmonic potential, i.e.,  $\omega_0 \neq 0$ . The reduced density matrix of the damped harmonic oscillator is given by [6]

$$\langle x | \rho_A | x' \rangle = \sqrt{\frac{4b}{\pi}} e^{-a(x-x')^2 - b(x+x')^2} \quad (10)$$

$$a = \frac{\langle p^2 \rangle}{2} \quad , \quad b = \frac{1}{8\langle q^2 \rangle} \quad .$$

The above expression is deduced such that the correct variances for position and momentum are obtained. At  $T = 0$  the expectation values are given by

$$\langle q^2 \rangle = \frac{1}{2\omega_0} f(\kappa) \quad (11)$$

$$\langle p^2 \rangle = \omega_0^2 (1 - 2\kappa^2) \langle q^2 \rangle + \frac{2\omega_0 \kappa}{\pi} \ln \left( \frac{\omega_c}{\omega_0} \right) \quad (12)$$

with  $\kappa = \eta/2\omega_0$  and

$$f(\kappa) = \frac{1}{\pi} \frac{\ln[(\kappa + \sqrt{\kappa^2 - 1})/(\kappa - \sqrt{\kappa^2 - 1})]}{\sqrt{\kappa^2 - 1}} \quad (13)$$

The parameter  $\kappa$  represents the friction parameter and the system experiences a crossover from coherent to incoherent oscillations for  $\kappa = 1$ .

Taylor expanding the logarithm of the entropy, Eq. (6), we need to evaluate the general  $n$ -dimensional integral

$$\int_{-\infty}^{\infty} dx_1 \dots dx_n \exp\left(-\sum_{i,j=1}^n x_i A_{i,j} x_j\right) = \frac{\pi^{n/2}}{\sqrt{\det A}} \quad (14)$$

where  $A$  is given by the translationally invariant tight-binding matrix with  $A_{i,i} = 2(a+b)$ ,  $A_{i+1,i} = A_{i,i+1} = -(a-b)$  ( $n+1 \equiv 1$ ) and zero otherwise. The determinant of the matrix is given by its eigenvalues and reads

$$\det A = (2a)^n (1 - b/a)^n \prod_{m=1}^n \left[1 + \frac{2b}{a-b} - \cos k_m\right] \quad (15)$$

with  $k_m = 2\pi m/n$ . Considering the  $n$ -dimensional translationally invariant, but non-hermitian matrix  $\tilde{A}_{i,i} = 1$ ,  $\tilde{A}_{i+1,i} = 1 - \varepsilon$  ( $n+1 \equiv 1$ ) and zero otherwise, one obtains the following formula:

$$\prod_{m=1}^n \left[1 + \frac{\varepsilon^2}{2(1-\varepsilon)} - \cos k_m\right] = \frac{(1 - (1-\varepsilon)^n)^2}{2^n (1-\varepsilon)^n} \quad (16)$$

For  $\omega_c/\omega_0 \gg 1$ , we have

$$\begin{aligned} a/b &= 4\langle q^2 \rangle \langle p^2 \rangle \\ &= f(\kappa) \left[ (1 - 2\kappa^2) f(\kappa) + \frac{4\kappa}{\pi} \ln\left(\frac{\omega_c}{\omega_0}\right) \right] \gg 1. \end{aligned} \quad (17)$$

In this limit, we can thus set  $\varepsilon^2 = 4b/a \ll 1$  and the  $n$ -dimensional integral can be approximated to yield

$$\int dx \langle x | \rho_A^n | x \rangle \rightarrow \frac{\tilde{\varepsilon}^n}{1 - (1-\varepsilon)^n}, \quad (18)$$

with  $\tilde{\varepsilon} \equiv \varepsilon\sqrt{1-\varepsilon}/\sqrt{1-\varepsilon^2/4}$ . Expanding the denominator as geometrical series, we have for the entropy

$$S = -\left(\frac{\tilde{\varepsilon}}{\varepsilon} \ln \tilde{\varepsilon} + \frac{\tilde{\varepsilon}}{\varepsilon^2} \ln(1-\varepsilon)\right). \quad (19)$$

In the limit  $\varepsilon \approx \tilde{\varepsilon} \ll 1$ , the leading behavior of the entropy is given by  $S \sim \ln(a/b)$ . We thus find a non-analyticity at  $\kappa = 1$ , the point of the crossover of incoherent to coherent oscillations. The leading behavior of Eq. (19) is plotted in Fig. 1 as function of the dimensionless coupling strength  $\alpha = q_0^2 \eta / (2\pi)$  with the characteristic length scale  $q_0 = 1/\sqrt{\omega_0}$  for  $\omega_c/\omega_0 = 100$  (full line). In the inset, the non-analyticity of the derivative of the entropy with respect to the coupling strength  $S' \equiv \partial_\alpha S$  at  $\alpha = 1/\pi$  can be seen.

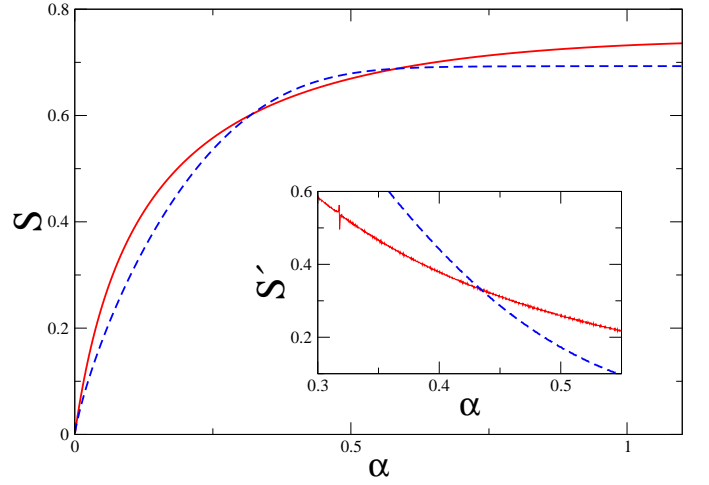


FIG. 1: (Color online). The entropy  $S$  of the dissipative oscillator (full) and the dissipative two-level system (dashed) with Ohmic coupling as function of the dimensionless coupling strength  $\alpha$ . Inset: The derivative of the entropy  $S'$  with respect to the coupling strength  $\alpha$  is shown as function of  $\alpha$ .

### III. SPIN-BOSON MODEL

A prominent dissipative model is given by the spin-boson model or dissipative two-level system (TLS). The Hamiltonian without bias reads

$$H = \frac{\Delta_0}{2} \sigma_x + \sum_k \omega_k b_k^\dagger b_k + \sigma_z \sum_k \frac{\lambda_k}{2} (b_k + b_k^\dagger) \quad (20)$$

The operators  $b_k^{(\dagger)}$  resemble the bath degrees of freedom and  $\sigma_x, \sigma_y, \sigma_z$  denote the Pauli spin matrices. They obey the canonical commutation relations and the spin-1/2 algebra, respectively.

The coupling constants  $\lambda_k$  are parameterized by the spectral function

$$J(\omega) = \sum_k \lambda_k^2 \delta(\omega - \omega_k). \quad (21)$$

In the relevant low-energy regime, the spectral function is generally parameterized as a power-law, i.e.,  $J(\omega) \propto 2\alpha\omega^s \Lambda_0^{1-s}$  where  $\alpha$  denotes the coupling constant,  $s$  the bath type and  $\Lambda_0$  the cutoff-frequency.

The general reduced density matrix of the spin-boson model is given by

$$\rho_A = \frac{1}{2} \begin{pmatrix} 1 + \langle \sigma_z \rangle & \langle \sigma_x \rangle \\ \langle \sigma_x \rangle & 1 - \langle \sigma_z \rangle \end{pmatrix}. \quad (22)$$

Since there is no symmetry breaking field in the above Hamiltonian, we have  $\langle \sigma_z \rangle = 0$ . The eigenvalues are thus given by  $\lambda_\pm = (1 \pm \langle \sigma_x \rangle)/2$  and the entropy reads

$$S = -\frac{1}{2} \left[ \ln((1 - \langle \sigma_x \rangle)^2 / 4) + \langle \sigma_x \rangle \ln\left(\frac{1 + \langle \sigma_x \rangle}{1 - \langle \sigma_x \rangle}\right) \right]. \quad (23)$$

The value of  $\langle \sigma_x \rangle$ , at zero temperature, is given by

$$\langle \sigma_x \rangle = 2 \frac{\partial E}{\partial \Delta_0} \quad (24)$$

where  $E$  is the energy of the ground-state. To obtain the ground-state energy, a scaling analysis for the free energy at arbitrary temperature is considered, see the appendix. In the following, we use this approach to calculate  $E(\Delta_0)$  and  $\langle \sigma_x \rangle$  which will set the basis of our discussion on the entanglement properties of the spin-boson model.

### A. Ohmic dissipation

In the Ohmic case ( $s = 1$ ), there is a phase transition at zero temperature at the critical coupling strength  $\alpha = 1$  [28, 29]. The transition is also manifested in the renormalized tunnel element  $\Delta_{\text{ren}}$ , i.e.,  $\Delta_{\text{ren}} = \Delta_0(\Delta_0/\Lambda_0)^{\alpha/(1-\alpha)}$  for  $\alpha < 1$  and  $\Delta_{\text{ren}} = 0$  for  $\alpha > 1$ .

The free energy is determined by (see the appendix)

$$F = \int_{\Delta_{\text{ren}}}^{\Lambda_0} \left( \frac{\Delta(\Lambda)}{\Lambda} \right)^2 d\Lambda. \quad (25)$$

The ground state energy  $E$  can then be written as

$$E = \begin{cases} \frac{C}{1-2\alpha} \left[ \Delta_0 \left( \frac{\Delta_0}{\Lambda_0} \right)^{\frac{\alpha}{1-\alpha}} - \frac{\Delta_0^2}{\Lambda_0} \right] & 0 < \alpha < \frac{1}{2} \\ 2C \frac{\Delta_0^2}{\Lambda_0} \log \left( \frac{\Delta_0}{\Lambda_0} \right) & \alpha = \frac{1}{2} \\ \frac{C}{2\alpha-1} \left[ \frac{\Delta_0^2}{\Lambda_0} - \Delta_0 \left( \frac{\Delta_0}{\Lambda_0} \right)^{\frac{\alpha}{1-\alpha}} \right] & \frac{1}{2} < \alpha < 1 \\ C \frac{\Delta_0^2}{\Lambda_0} & \alpha > 1 \end{cases} \quad (26)$$

where  $C$  is a numerical constant. For  $\alpha = 1/2 \pm \epsilon$ , we have

$$\frac{d \ln \langle \sigma_x \rangle}{d\alpha} \Big|_{\alpha=1/2 \pm \epsilon} \propto \frac{1}{\epsilon}. \quad (27)$$

For  $\alpha = 1 - \epsilon$ , we have

$$\frac{d \ln \langle \sigma_x \rangle}{d\alpha} \Big|_{\alpha=1-\epsilon} \propto \ln \left( \frac{\Delta_0}{\Lambda_0} \right) \quad (28)$$

The non-analyticity around  $\alpha = 1$  is thus far weaker than around  $\alpha = 1/2$ . This non-analyticity is also present in the entropy as can be seen from the expression Eq. (22).

The entropy  $S$  of the dissipative two-level system with Ohmic coupling is plotted in Fig. 1 as function of the dimensionless coupling strength  $\alpha$  for  $\omega_c/\Delta_0 = 100$  (dashed line). The inset shows the derivative of the entropy with respect to the coupling strength. The entropy quickly saturates after the transition from coherent to incoherent oscillations at  $\alpha = 1/2$ , but the non-analyticity of Eq. (27) cannot be seen on this scale.

### B. Non-Ohmic dissipation

) The calculation of  $E(\Delta_0)$  and  $\langle \sigma_x \rangle$  can be extended to the spin-boson model with non-Ohmic dissipation ( $s \neq 1$ ). In general, the dependence of the effective tunneling term on the cutoff,  $\Delta(\Lambda)$  is:

$$\Delta(\Lambda) = \Delta_0 \exp \left( -\frac{1}{2} \int_{\Lambda}^{\Lambda_0} \frac{J(\omega)}{\omega^2} d\omega \right) \quad (29)$$

with the spectral function given in Eq. (21). A renormalized low energy term,  $\Delta_{\text{ren}}$ , can be defined by

$$\Delta_{\text{ren}} = \Delta_0 e^{-\int_{\Delta_{\text{ren}}}^{\Lambda_0} \frac{J(\omega)}{\omega^2} d\omega}. \quad (30)$$

The free energy is again determined by Eq. (25), though cannot be evaluated analytically, anymore. The scaling behavior of the renormalized tunneling given in Eq. (29) is no longer a power law, as in the Ohmic case. Still, we can distinguish two limits:

i) The renormalization of  $\Delta(\Lambda)$  is slow. In this case, the integral in Eq. (25) is dominated by the region  $\Lambda \sim \Lambda_0$ , where the function in the integrand goes as  $\Lambda^{-2}$ . The integral is dominated by its high cutoff,  $\Lambda_0$ , and the contribution from the region near the lower cutoff,  $\Delta_{\text{ren}}$ , can be neglected. Then, we obtain that  $F(\Delta_0) \sim \Delta_0^2/\Lambda_0$ .

ii) The renormalization of  $\Delta(\Lambda)$  is fast. In this case, the contribution to the integral in Eq. (25) from the region  $\Lambda \approx \Lambda_0$  is small. The value of the integral is dominated by the region near  $\Lambda \simeq \Delta_{\text{ren}}$ . As  $\Delta_{\text{ren}}$  is the only quantity with dimensions of energy needed to describe the properties of the system in this range, we expect that  $F(\Delta_0) \approx \Delta_{\text{ren}}$ .

In the scaling limit,  $\Delta_0/\Lambda_0 \ll 1$ , the values of the two terms,  $\Delta_{\text{ren}}$  and  $\Delta_0^2/\Lambda_0$ , become very different. In addition, there are no other energy scales which can qualitatively modify the properties of the system. We thus conclude that only the two terms mentioned above will contribute to the free energy. Hence, we can write:

$$F(\Delta_0) \simeq \max \left( \Delta_{\text{ren}}, \frac{\Delta_0^2}{\Lambda_0} \right) \quad (31)$$

The above equation is now used to discuss the possible transition between underdamped to overdamped oscillations for non-Ohmic environments. Notice that it also applies for Ohmic baths.

#### 1. Super-Ohmic dissipation

In the super-Ohmic case ( $s > 1$ ), Eq. (30) always has a solution and, moreover, we can also set the lower limit of the integral to zero. This yields

$$\Delta_{\text{ren}} = \Delta_0 e^{-\int_0^{\Lambda_0} \frac{J(\omega)}{\omega^2} d\omega} \approx \Delta_0 e^{-\alpha/(s-1)}. \quad (32)$$

For  $\alpha \gg 1$  we have  $\Delta_{\text{ren}} \ll \Delta_0$ , but there is no transition from localized to delocalized behavior.

Using Eq. (31) in the super-Ohmic case  $s > 1$ , we can approximately write:

$$\langle \sigma_x \rangle \simeq \max \left( e^{-\alpha/(s-1)}, \frac{\Delta_0}{\Lambda_0} \right) \quad (33)$$

We thus find a transition from underdamped to overdamped oscillations at some critical coupling strength  $\alpha \sim (s-1) \log(\Lambda_0/\Delta_0)$ .

It is finally interesting to note that the scaling analysis discussed in Ref. [30] is equivalent to the scheme used here.

## 2. Sub-Ohmic dissipation

In the sub-Ohmic case ( $s < 1$ ), it is not guaranteed that Eq. (30) has a solution. In general, a solution only exists when  $\Delta_0/\Lambda_0$  is not much smaller than 1.

The existence of a phase transition in case of a sub-Ohmic bath was first proved in Ref. [31]. Whereas the relation in Eq. (30) and a similar analysis based on flow equations for Hamiltonians [32] yields a discontinuous transition between the localized and delocalized regimes, detailed numerical calculations suggest that the transition is continuous [33].

Since there is a phase transition from localized to non-localized behavior, there might also be a transition between overdamped to underdamped oscillation. In Ref. [34], this transition was discussed on the basis of spectral functions analogous to the discussion of Ref. [35, 36] for Ohmic dissipation. It was found that for  $s > 0.5$  the transition takes place for lower values of  $\alpha$  as in the Ohmic case, e.g., for  $s = 0.8$  and  $\Lambda_0/\Delta_0 = 10$  the transition coupling strength is  $\alpha^* \approx 0.2$ .

Using Eq. (30) and Eq. (31) yields for the sub-Ohmic case:

$$\langle \sigma_x \rangle \simeq \begin{cases} 1 & \text{delocalized regime} & \frac{\Delta_0}{\Lambda_0} \simeq 1 \\ \frac{\Delta_0}{\Lambda_0} & \text{localized regime} & \frac{\Delta_0}{\Lambda_0} \ll 1 \end{cases} \quad (34)$$

The analysis used in the previous cases leads us to expect coherent oscillations in the delocalized regime.

We can extend the study of the sub-Ohmic case to the vicinity of the second order transition described in Ref. [37], which in our notation takes place for  $\alpha = s\Delta_0/\Lambda_0 \ll 1$ . In this regime, which cannot be studied using the Franck-Condon like renormalization in Eq. (30), we use the renormalization scheme around the fully coherent state proposed in Ref. [37]. To one-loop order, the beta-function for the dimensionless quantity (expressed in our notation)  $\tilde{\kappa} = (\alpha\Lambda)/\Delta$  then reads

$$\beta(\tilde{\kappa}) = -s\tilde{\kappa} + \tilde{\kappa}^2. \quad (35)$$

Near the transition, in the delocalized phase,  $\tilde{\kappa}$  thus scales towards zero as

$$\tilde{\kappa}(\Lambda) = \tilde{\kappa}_0 \left( \frac{\Lambda}{\Lambda_0} \right)^s. \quad (36)$$

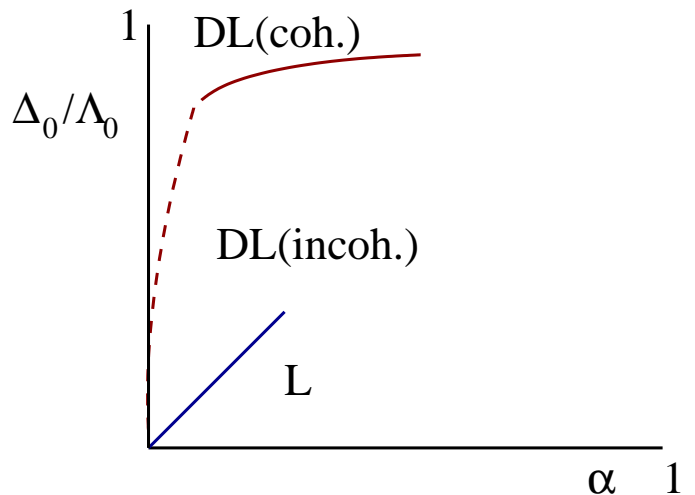


FIG. 2: (Color online). Schematic picture of the different regimes in the sub-Ohmic dissipative TLS studied in the text. DL stands for delocalized phase, while L denotes a localized phase. The lower blue line denotes the continuous transition studied in Ref. [37]. The red line marks the boundaries of a regime characterized by a small renormalization of the tunneling rate, Eq. (30), and coherent oscillations.

The scaling of  $\langle \sigma_x \rangle$  is

$$\frac{\partial \langle \sigma_x \rangle}{\partial \Lambda} = -\tilde{\kappa}(\Lambda) \frac{\Delta}{\Lambda^2}. \quad (37)$$

The fact that the scheme assumes a fully coherent state as a starting point implies that  $\Delta$  is not renormalized. Inserting Eq. (36) into Eq. (37), we find:

$$\frac{\partial \langle \sigma_x \rangle}{\partial \Lambda} = -\tilde{\kappa}_0 \left( \frac{\Lambda}{\Lambda_0} \right)^s \frac{\Delta_0}{\Lambda^2} \quad (38)$$

If we calculate  $\langle \sigma_x \rangle$  from this equation, we find that the resulting integral diverges as  $\Lambda \rightarrow 0$  for  $s \leq 1$ . This result implies that  $\langle \sigma_x \rangle \ll 1$ . For sufficiently low values of the effective cutoff,  $\Lambda$ , the value of  $\langle \sigma_x \rangle$  can be calculated using a perturbation expansion on  $\Delta_0$ , leading to  $\langle \sigma_x \rangle \sim \Delta_0/\Lambda_0$ . This result implies the absence of coherent oscillations, as in the similar cases discussed previously.

A schematic picture of the regimes studied for the sub-Ohmic TLS is shown in Fig. [2].

## IV. SUMMARY

In this article, the entanglement properties of dissipative systems were investigated on the basis of the von Neumann entropy.

We first investigated two integrable dissipative quantum systems -the free dissipative particle and the dissipative harmonic oscillator - and calculated the von Neumann entropy. In the former case, this could be done

exactly and no non-analyticity was found. The case of the harmonic oscillator is the more interesting one since it exhibits a transition from underdamped to overdamped oscillations for increasing dissipation. This transition is also manifested in the entropy, or equivalently in the entanglement which was calculated in the limit of large bath cutoff.

We also calculated the von Neumann entropy for the spin-boson model on the basis of a scaling equation for the free energy. Only in the Ohmic case, the resulting integral could be evaluated and we analyzed the non-analyticity at the transition from underdamped to overdamped oscillations. We found that the non-analyticity more pronounced than at the actual phase transition.

In the non-Ohmic case, we argued that the transition between coherent and decoherent oscillation takes place when the value of  $\langle \sigma_x \rangle$  becomes comparable to the result obtained using a perturbation expansion in the tunneling matrix,  $\Delta$  (as is the case for Ohmic dissipation). In the super-Ohmic case, this always yields a critical coupling strength at zero temperature which differs from the analysis in Ref. [38].

In the sub-Ohmic case, the scaling approach can only be trusted when the tunnel matrix element is of the same order of magnitude as the cutoff. Then a transition between coherent to non-coherent oscillations is possible before the system becomes localized. For the regime where the cutoff represents the largest energy scale, we applied a novel renormalization scheme proposed in Ref. [37]. We find that, in the delocalized phase, the system is most likely incoherent.

Concerning the entanglement properties for the non-Ohmic case, we were not able to discuss possible non-analyticities since the regime is analytically not accessible. Numerical work in this direction is planned for the future.

To conclude, we suppose that entanglement properties are closely connected to the transition of coherent to incoherent tunneling. Our observations might be useful for future quantum bit manipulations.

## V. ACKNOWLEDGMENTS

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## APPENDIX A: CALCULATION OF THE FREE ENERGY OF THE DISSIPATIVE TLS

We calculate the free energy of the dissipative two level system following the scaling approach discussed for the Kondo problem in Refs. [12, 39], and formulated in a more general way in Ref. [40]. For the general long-ranged Ising model, the scaling approach was first applied by Kosterlitz [30].

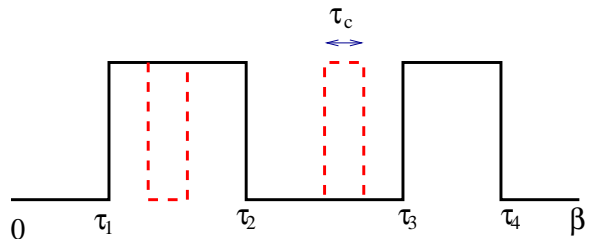


FIG. 3: (Color online). Sketch of the instanton pairs which renormalizes the calculation of the free energy of the dissipative TLS.

The partition function of the model can be expanded in powers of  $\Delta^2$  as

$$Z = \sum_n \frac{\Delta^{2n}}{2n!} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_{2n} \prod_{ij=1, \dots, 2n} f[(\tau_i - \tau_j)/\tau_c] \quad (\text{A1})$$

where  $f[(\tau_i - \tau_j)/\tau_c]$  denotes the interaction between the kinks located at positions  $\tau_i$  and  $\tau_j$ . A term in the series is schematically depicted in Fig. [3]. The scaling procedure lowers the short time cutoff of the theory from  $\tau_c$  to  $\tau_c - d\tau_c$ . This process removes from each term in the sum in Eq. (A1) details at times shorter than  $\tau_c - d\tau_c$ . The rescaling  $\tau_c \rightarrow \tau_c - d\tau_c$  implies the change  $\Delta \rightarrow \Delta(1 + d\tau_c/\tau_c)$ . The dependence of  $f[(\tau_i - \tau_j)/\tau_c]$  leads to another rescaling, which can be included in a global renormalization of  $\Delta$  [12, 39, 40]. In addition, configurations with an instanton-antiinstanton pair at distances between  $\tau_c$  and  $\tau_c - d\tau_c$  have to be replaced by configurations where this pair is absent, as schematically shown in Fig. [3]. The number of removed pairs is proportional to  $d\tau_c/\tau_c$ . The center of the pair can be anywhere in the interval  $0 \leq \tau \leq \beta$ . The final effect is the rescaling:

$$Z \rightarrow Z(1 + \Delta^2 \beta d\tau_c) \quad (\text{A2})$$

Writing  $Z$  as  $Z = e^{-\beta F}$ , where  $F$  is the free energy, Eq. (A2) can be written as:

$$-\frac{\partial F}{\partial \tau_c} = \Delta^2(\tau_c) \quad (\text{A3})$$

In the Ohmic case, the dependence of  $\Delta$  on  $\tau_c = \Lambda^{-1}$  is

$$\Delta(\Lambda) = \Delta_0 \left( \frac{\Lambda}{\Lambda_0} \right)^\alpha \quad (\text{A4})$$

and, finally, we find the following relation:

$$\frac{\partial F}{\partial \Lambda} = \left[ \frac{\Delta(\Lambda)}{\Lambda} \right]^2 = \left( \frac{\Delta_0}{\Lambda_0} \right)^2 \left( \frac{\Lambda}{\Lambda_0} \right)^{2\alpha-2} \quad (\text{A5})$$

This equation ceases to be valid for  $\Lambda \simeq \Delta_{\text{ren}}$ . For finite temperatures, we obtain

$$F(T) = \int_T^{\Lambda_0} \frac{\partial F}{\partial \Lambda} d\Lambda. \quad (\text{A6})$$

It is interesting to apply this analysis to a free two level system. The value of  $\Delta_0$  does not change under scaling. We find the following expression:

$$\frac{\partial F}{\partial \Lambda} = \begin{cases} \left(\frac{\Delta_0}{\Lambda}\right)^2 & \Delta_0 \ll \Lambda \\ 0 & \Lambda \ll \Delta_0 \end{cases} \quad (\text{A7})$$

Inserting this expression into Eq. (A6), we obtain

$$F(T) = \begin{cases} \frac{\Delta_0^2}{T} & \Delta_0 \ll T \\ \Delta_0 & T \ll \Delta_0 \end{cases} \quad (\text{A8})$$

and, finally:

$$\langle \sigma_x \rangle = \frac{\partial F}{\partial \Delta_0} = \begin{cases} \frac{\Delta_0}{T} & \Delta_0 \ll T \\ 1 & T \ll \Delta_0 \end{cases} \quad (\text{A9})$$

in qualitative agreement with the exact result  $\langle \sigma_x \rangle = \tanh(\Delta_0/T)$ .

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